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Remarks on the Magnetic Top

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We revisit via a path-integral approach the magnetic top proposed recently by Barut, Božić, and Marić. We point out that the magnetic top has the SU(2) symmetry and that it can be viewed as a free top seen from a rotating frame. We present an alternative path-integral quantization of the magnetic top on the basis of the symmetry, and show that the magnetic coupling does not participate in altering the spin quantum numbers.

1. INTRODUCTION

To every classical observable there corresponds a quantum operator which is an operator-valued function of the fundamental observables, **r** and **p**. According to the correspondence principle, a quantum mechanical operator must have a classical counterpart. However, spin is a well-known exception. Although the quantum mechanical spin may be visualized as the classical spinning of a particle, the spin operator has no classical limit in the sense of the correspondence principle.

Historically, since Pauli(1) introduced the double-valued internal degree of freedom of an electron in 1925, a number of physicists supplied classical interpretations for the new degree. Notably, Uhlenbeck and Goudsmit(2) interpreted it as the angular momentum due to spinning of an electron and established the concept of spin in quantum mechanics even though the classical interpretation is not without difficulty.(3) Pauli(4) showed that the eigenvalues of an angular momentum, if given by $\boldsymbol{l} = \mathbf{r} \times \mathbf{p}$, must be integers. The spin has remained as an essentially quantum

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mechanical concept. The double-valued spin property was understood from its birth in relation with the magnetic moment of the electron. As early as 1921, Compton[5] suggested that the electron would be a magnetic elementary particle spinning like a tiny gyroscope. Mechanically, a gyroscope is equivalent to a spinning spherical top. The first quantum mechanical treatment of a symmetric top by Reiche[6] in 1926 did not reveal the spin concept. However, in 1931, Casimir[7] discovered that the angular momentum of a spherical top could be an integer or a half integer. Although we often associate the integral spins with the single-valued representation of the rotation group $SO(3)$ and the half-integral spins with the double-valued representation of $SO(3)$, rigorously speaking, a representation of a group has to be single-valued. The representation of $SU(2)$ can accommodate spins of a half integer as well as those of an integer. In this regard, Reiche's top was of $SO(3)$, whereas Casimir's was of $SU(2)$.

The lack of a classical description of spin was a serious problem for Feynman's path-integral approach to quantum mechanics.[8] The original path integral formulation, as was originally intended,[9] is to deal solely with classical variables. It is more than desirable that there is a classical source from which the path integral quantization can generate the spin property. In order to incorporate spin into Feynman's path integral, Schulman[10] asserted along a line similar to Casimir's that the spin property was in the spinning top of $SU(2)$ as its quantization brings about both integral and half-integral eigenvalues.

There are other interesting attempts[11, 12] to locate the classical source of spin from quantization of a particle confined to move on the sphere S^3. These are different but not completely independent from quantization of a top. The sphere S^3 on which a particle is quantized is indeed the $SU(2)$ group manifold.

Recently, Barut, Božić, and Marić[13] proposed a magnetic top as a classical model for spin. They have shown by quantizing the magnetic top with a magnetic moment placed in a uniform magnetic field that it results in Pauli's theory of spin. Barut and Duru[14] also used a path integral to show that quantization of the magnetic top leads to integral and half-integral spins. In the present paper, once again, we examine the magnetic top from a path integral point of view. First, contrary to what has previously been suggested in Ref. 14, we show that $SU(2)$ symmetry is not broken; but the magnetic top has $SU(2)$ symmetry as a free spherical top does. Secondly, taking advantage of the symmetry, we quantize it by path integration in a way different from those in previous investigations.[13, 14] Then we point out that from the path integral point of view the magnetic top is no more special than a free spherical top in generating integral and half-integral spins.

2. $SU(2)$ SYMMETRY OF THE MAGNETIC TOP

The magnetic top is defined as a spherically symmetric top which carries a magnetic moment proportional to the kinetic angular momentum, $\mathbf{M} = g\mathbf{L}$. Here g is the gyromagnetic ratio. Since a symmetric top spinning with angular frequency $\boldsymbol{\omega}$ has the angular momentum $\mathbf{L} = I\boldsymbol{\omega}$, where I is the top's moment of inertia, the magnetic moment may also be given by $\mathbf{M} = gI\boldsymbol{\omega}$. While the ordinary massive top spins in the gravitational field, the magnetic top couples to an external magnetic field. In a uniform magnetic field $\mathbf{B}$, the Lagrangian of the top is given by

$$L = \tfrac{1}{2} I \boldsymbol{\omega}^2 + \mathbf{M} \cdot \mathbf{B} \tag{1}$$

In a recent paper,[14] it has been suggested that the well-known $SU(2)$ symmetry of a free spherical top is broken in the magnetic top. Contrary to such an observation, however, we can show that the $SU(2)$ symmetry survives as a healthy symmetry of the magnetic top in a uniform magnetic field. The Lagrangian (1) characterizes the magnetic top seen from the coordinate frame fixed in space. If we choose a rotating frame in which the angular frequency takes the form $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} + g\mathbf{B}$, then the Lagrangian may be expressed as

$$L = \tfrac{1}{2} I \tilde{\boldsymbol{\omega}}^2 - \tfrac{1}{2} I g^2 B^2 \tag{2}$$

Since the last term is a constant, the Lagrangian (2) is basically the same as that of a free top with the angular momentum $\tilde{\mathbf{L}} = \mathbf{L} + Ig\mathbf{B}$. Therefore, the $SU(2)$ symmetry of the free top should be carried over to the magnetic top.

To be more explicit, choosing the magnetic field pointing to the direction of the z-axis, i.e., $\mathbf{B} = B\mathbf{e}_3$, we may write the Lagrangian (1) in terms of Euler angles (ϕ, θ, ψ) as[15]

$$L = \tfrac{1}{2} I(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\phi}\dot{\psi}\cos\theta) + gIB(\dot{\phi} + \dot{\psi}\cos\theta) \tag{3}$$

Then, applying the transformation

$$\dot{\phi} \to \dot{\varphi} = \dot{\phi} + gB \tag{4}$$

we can reduce the Lagrangian (3) into

$$L = \tilde{L} - \tfrac{1}{2} L I g^2 B^2 \tag{5}$$

where

$$\tilde{L} = \tfrac{1}{2} I(\dot{\theta}^2 + \dot{\varphi}^2 + \dot{\psi}^2 + 2\dot{\varphi}\dot{\psi}\cos\theta) \tag{6}$$

Since the transformation (4) is integrable, the Lagrangian (5) expressed in terms of the new set of Euler angles (φ, θ, ψ) is identical in form with the Lagrangian for a free spherical top except for the last term which gives rise only to a constant shift in energy. The Lagrangian $\tilde{L}$ is an invariant under $SU(2)$ as well as $SO(3)$.

According to Feynman's prescription, the propagator is given as a path integral, that is, an infinite convolution of short-time propagators. For the Lagrangian (5),

$$K(\varphi'', \theta'', \psi''; \varphi', \theta', \psi'; \tau) = \lim_{N \to \infty} \int \prod_{j=1}^{N-1} d\mu(\varphi_j, \theta_j, \psi_j) \prod_{j=1}^{N} K(\varphi_j, \theta_j, \psi_k; \varphi_{j-1}, \theta_{j-1}, \psi_{j-1}; \varepsilon) \tag{7}$$

where $\varepsilon = t_j - t_{j-1} = \tau/N$ for all j and $d\mu(\varphi, \theta, \phi)$ is the normalized $SU(2)$-invariant measure. The jth short-time propagator has the form

$$K(\varphi_j, \theta_j, \psi_j; \varphi_{j-1}, \theta_{j-1}, \psi_{j-1}; \varepsilon) = A_j e^{-iS_j/h} \tag{8}$$

where A_j is an amplitude to be determined by the initial condition,

$$\lim_{\varepsilon \to 0} K(\varphi, \theta, \psi; \varphi', \theta', \psi'; \varepsilon) = \left| \frac{d\varphi\, d\theta\, d\psi}{d\mu(\varphi, \theta, \psi)} \right| \delta(\varphi - \varphi')\, \delta(\theta - \theta')\, \delta(\psi - \psi') \tag{9}$$

and

$$S_j = \int_{t_{j-1}}^{t_j} \left\{ \tilde{L} - \frac{1}{2} I g^2 B^2 + C_0 \right\} dt \tag{10}$$

is the action for a short time interval ε. A constant C_0 is added to the Lagrangian, which corresponds to the so-called quantum correction term. In order to obtain the standard energy spectrum for the free top in the absence of the external field, we have to choose $C_0 = 3h^2/(32I)$ (see Ref. 17). The short-time action (10) may be expressed in the form

$$S_j \doteq \frac{4I}{\varepsilon} \left\{ 1 - \cos\left(\frac{1}{2} \Omega_j \right) \right\} + C\varepsilon \tag{11}$$

where $C = C_0 - Ig^2B^2/2$, and

$$\cos\left(\frac{1}{2}\Omega_j\right) = \cos\frac{\theta_j}{2} \cos\frac{\theta_{j-1}}{2} \cos\frac{(\varphi_j - \varphi_{j-1}) + (\psi_j - \psi_{j-1})}{2} + \sin\frac{\theta_j}{2} \sin\frac{\theta_{j-1}}{2} \cos\frac{(\varphi_j - \varphi_{j-1}) - (\psi_j - \psi_{j-1})}{2} \tag{12}$$

In fact, as ε tends to zero, $(S_j/\varepsilon) - C$ approaches the Lagrangian (6).

At this point, we examine the $SU(2)$ symmetry of the propagator (7) with the action (10). For convenience, let us represent a group element of $SU(2)$ in terms of the new set of Euler angles (φ, θ, ψ) as

$$g(\varphi, \theta, \psi) = \begin{pmatrix} \cos\frac{\theta}{2} \exp\left(i\frac{\varphi + \psi}{2}\right) & i\sin\frac{\theta}{2} \exp\left(i\frac{\varphi - \psi}{2}\right) \\ i\sin\frac{\theta}{2} \exp\left(-i\frac{\varphi - \psi}{2}\right) & \cos\frac{\theta}{2} \exp\left(-i\frac{\varphi + \psi}{2}\right) \end{pmatrix} \tag{13}$$

Let J_k $(k = 1, 2, 3)$ be the generators of the $SU(2)$ group satisfying the Lie algebra $su(2)$,

$$[J_i, J_j] = iJ_k \qquad (i, j, k \text{ in cyclic order}) \tag{14}$$

They can be represented in terms of the Pauli matrices as $J_k = \sigma_k/2$. Then we may easily put the group element (13) into the form

$$g(\varphi, \theta, \psi) = \exp(i\varphi J_3) \exp(i\theta J_1) \exp(i\psi J_3) \tag{15}$$

Thus we can establish homeomorphism between (13) and (15).

As is well known, if J and m denote the eigenvalues $J(J+1)$ of the Casimir operator $Q = J_1^2 + J_2^2 + J_3^2$ and those of J_3, respectively, then unitary irreducible representations of $SU(2)$ are given by

$$\begin{aligned} A_J&: \quad J = 0, 1, 2, \ldots; \qquad m = 0, \pm 1, \pm 2, \ldots \pm J \\ B_J&: \quad J = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots; \qquad m = \pm\tfrac{1}{2}, \pm\tfrac{3}{2}, \pm\tfrac{5}{2}, \ldots \pm J \end{aligned} \tag{16}$$

The representation space is the Hilbert space spanned by vectors $|J, m\rangle$. The $SU(2)$ group is locally isomorphic to the rotation group $SO(3)$, that is, the generators of the two groups obey the same Lie algebra (14). While the group manifold of the former is simply connected, that of the latter is doubly connected. The A_J and B_J in the above are often referred to as the single-valued and double-valued representations of $SO(3)$, respectively. To span the representation space of $SU(2)$, both A_J and B_J are needed.

Using the $SU(2)$ group element (13), we can also verify

$$\mathrm{Tr}(g_j g_{j-1}^{-1}) = 2\cos(\tfrac{1}{2}\Omega_j) \tag{17}$$

where $g_j = g(\varphi_j, \theta_j, \psi_j)$ and $g_{j-1} = g(\varphi_{j-1}, \theta_{j-1}, \psi_{j-1})$, both of which belong to $SU(2)$. Hence the short-time action (11) takes on the form[18,19]

$$S_j = \frac{4I}{\varepsilon}\left\{1 - \frac{1}{2}\mathrm{Tr}(g_j g_{j-1}^{-1})\right\} - \frac{1}{2} I g^2 B^2 \varepsilon + C_0 \varepsilon \tag{18}$$

The corresponding short-time propagator (8) becomes a function of $\mathrm{Tr}(g_j g_{j-1}^{-1})$.

Let h be any element of $SU(2)$. Under the transformation $g \to g' = hgh^{-1}$, the trace in the short-time action (18) remains unchanged and so does the short-time propagator. As far as the measure $d\mu(\varphi, \theta, \psi)$ is chosen invariant, the propagator (7) is an invariant under any $SU(2)$ group action.

The transformation (4) for rotating the space frame is integrable, leading to a time-dependent transformation of the angular variable ϕ:

$$\phi \to \varphi = \phi + \alpha(t) \tag{19}$$

where $\alpha(t) = gB(t - t') + \varphi' - \phi'$. The inverse transformation of (19) induces a group action

$$h_\alpha = \exp(-i\alpha(t) J_3) \tag{20}$$

on the Hilbert space where the group elements g of $SU(2)$ act. Obviously, this belongs to $SU(2)$ under which tithe propagator remains invariant. Therefore, the magnetic top has $SU(2)$ symmetry.

3. PATH-INTEGRAL QUANTIZATION

Now that $SU(2)$ symmetry is good for the magnetic top, let us take advantage of the symmetry to quantize the top. As Barut and Duru[14] did, we quantize the system by using Feynman's path integral. While they used the Hamiltonian path integral, we calculate the Lagrangian path integral (7) with symmetry consideration.

Since the short-time propagator (8) is a function of $SU(2)$ group elements, we wish to carry out the path integral on the $SU(2)$ manifold. Namely, we cast the path integral into the form

$$K(\varphi'', \theta'', \psi''; \varphi', \theta', \psi'; \tau) = \lim_{N\to\infty} \int_{SU(2)} \prod_{j=1}^{N-1} d\mu(g_j) \prod_{j=1}^{N} K(g_j, g_{j-1}; \varepsilon) \tag{21}$$

with the short-time propagator

$$K(g_j, g_{j-1}; \varepsilon) = 2\pi^2 \left[\frac{4I}{2\pi i\hbar\varepsilon}\right]^{3/2} \exp\left[\frac{i}{\hbar} C\varepsilon\right] \exp\left[\frac{4iI}{\hbar\varepsilon}\left\{1 - \frac{1}{2}\mathrm{Tr}(g_j g_{j-1}^{-1})\right)\right\}\right] \tag{22}$$

Here, the invariant measure is to be normalized by

$$\int_{SU(2)} d\mu(g) = \frac{1}{16\pi^2}\int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\varphi \int_0^{4\pi} d\psi = 1 \tag{23}$$

To carry out the calculation, we use the character expansion formula[18] for $g, g' \in SU(2)$,

$$\exp\left[\frac{1}{2} z\, \mathrm{Tr}(g'g^{-1})\right] = \frac{2}{z}\sum_{2J=0}^{\infty} (2J+1)\, I_{2J+1}(z)\, \chi^J(g'g^{-1}) \tag{24}$$

where $I_\nu(z)$ is the modified Bessel function, and $\chi^J(g)$ is the character of the $(2J+1)$-dimensional irreducible representation of $SU(2)$ which is given by a Gegenbauer polynomial; for instance,

$$\chi^J(g_j g_{j-1}^{-1}) = C_{2J}^1\left(\cos\frac{\Omega_j}{2}\right) \tag{25}$$

The relation (24) is nothing but the well-known Gegenbauer formula[19]

$$\exp[z\cos\Theta] = \frac{2}{z}\sum_{2J=0}^{\infty} (2J+1)\, I_{2J+1}(z)\, C_{2J}^1(\cos\Theta) \tag{26}$$

The characters $\chi^J(g)$ of $SU(2)$ represented by the Gegenbauer polynomial $C_{2J}^1(\cos(\Omega/2))$ satisfy the relation

$$\int_{SU(2)} \chi^J(g_a g_b^{-1})\, \chi^{J'}(g_b g_c^{-1})\, d\mu(g_b) = \frac{1}{2J+1}\delta_{JJ'}\chi^J(g_a g_c^{-1}) \tag{27}$$

Therefore, integration of (21) on the $SU(2)$ manifold can be done very easily by using (27), the result being

$$K(g'', g'; \tau) = \sum_{2J=0}^{\infty} (2J+1)\, \kappa_J(\tau)\, \chi^J(g''g'^{-1}) \tag{28}$$

where

$$\kappa_J(\tau) = \lim_{N\to\infty} \prod_{j=1}^{N} \left[2\pi \left(\frac{2I}{i\pi h\varepsilon}\right)^{1/2} \exp\left(\frac{4iI}{h\varepsilon} + \frac{i}{h} C\varepsilon\right) I_{2J+1}\left(\frac{4I}{ih\varepsilon}\right)\right] \tag{29}$$

Since we can show that for $\mathrm{Re}\, z > 0$

$$\lim_{N\to\infty} \prod_{j=1}^{N} [2\pi z N/\tau]^{1/2}\, e^{-zN/\tau} I_\nu(zN/\tau) = \exp\left\{-\frac{\tau}{2z}\left(\nu^2 - \frac{1}{4}\right)\right\} \tag{30}$$

we get

$$\kappa_J(\tau) = \exp\left\{-\frac{ih\tau}{8I}\left[(2J+1)^2 - \frac{1}{4}\right] + \frac{i}{h} C\tau\right\} \tag{31}$$

In applying (30) to (29), we have assumed that I has a positive imaginary part which may be taken to be zero after calculation.[20]

As a consequence, we find the propagator in the form

$$K(g'', g'; \tau) = \sum_{2J=0}^{\infty} (2J+1)\, \chi^J(g''g'^{-1}) \exp\left\{-\frac{i\tau}{2Ih}[J(J+1)\, h^2 + I^2 g^2 B^2]\right\} \tag{32}$$

The character described by the Gegenbauer polynomial (25) can be expanded by means of the Wigner polynomials $d^J_{mn}(\theta) = \langle Jm|\exp[i\theta J_2]\,|Jn\rangle$ as

$$\chi^J(g''g'^{-1}) = \sum_{m,n=-J}^{J} e^{im(\varphi''-\varphi')} e^{in(\psi''-\psi')} d^J_{mn}(\theta'')\, d^{J*}_{mn}(\theta') \tag{33}$$

so the propagator may be decomposed as

$$\begin{aligned} K(\varphi'', \theta'', \psi''; \varphi', \theta', \psi'; \tau) &= \sum_{2J=0}^{\infty} (2J+1) \sum_{m,n=-J}^{J} e^{im(\varphi''-\varphi')} e^{in(\psi-\psi')} d^J_{mn}(\theta'')\, J^{J*}_{mn}(\theta') \\ &\quad \times \exp\left\{-\frac{i\tau}{2Ih}[J(J+1)\, h^2 + I^2 g^2 B^2]\right\} \end{aligned} \tag{34}$$

The Wigner polynomials used here satisfy the orthogonality relation (see, e.g., Ref. 19),

$$\int_0^\pi d^{J*}_{mn}(\theta)\, d^{J'}_{m'n'}(\theta) \sin\theta\, d\theta = \frac{2}{2J+1}\, \delta_{JJ'}\, \delta_{mm'}\, \delta_{nn'} \tag{35}$$

Transforming φ back to ϕ by (19) or by $\varphi'' - \varphi' = \phi'' - \phi' + gB\tau$, we can cast the propagator (34) into the form

$$\begin{aligned} K(\phi'', \theta'', \psi''; \phi', \theta', \psi'; \tau) &= \sum_{2J=0}^{\infty} {}' \sum_{m,n=-J}^{J} \Psi^*_{Jmn}(\phi', \psi', \theta')\, \Psi_{Jmn}(\phi'', \psi'', \theta'') \exp[-(i/h)\, \tau E_{Jm}] \end{aligned} \tag{36}$$

where

$$E_{Jm} = \frac{1}{2I} J(J+1)\, h^2 - gBmh + \frac{1}{2} I g^2 B^2 \tag{37}$$

and

$$\Psi_{Jmn}(\phi, \psi, \theta) = \sqrt{2J+1}\, e^{im\phi} e^{in\psi}\, d^J_{mn}(\theta) \tag{38}$$

Naturally we identify E_{Jm} of (37) and $\Psi_{Jmn}(\phi, \psi, \theta)$ of (38) with the energy spectrum and the energy eigenfunctions of the magnetic top normalized with the invariant measure $d\mu(g)$ of (23), respectively. Since we have used the irreducible representations (16) of $SU(2)$, the quantum number J assumes both integral and half-integral values; that is, we have $J = 0, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots$ on the basis of the $SU(2)$ symmetry. Thus, the eigenfunctions (38) represent both integral and half-integral spin states.

4. DOES THE MAGNETIC MOMENT CREATE SPINS?

Suppose the fiducial state of the magnetic top in the absence of the external magnetic field is given by a single-valued eigenfunction (38) with a set of integral quantum numbers (J, m, n). Can then an extra half-integral spin be created upon switching on the external magnetic field and activating the magnetic moment of the top? Although the energy spectrum is affected by the external field, the state function (38) remains the same regardless of whether the external field is switched on or not. The magnetic moment carried by the top does not participate in altering the quantum numbers. A free top of an integral spin will never acquire an additional half-integral spin via the magnetic coupling with an external field. The magnetic moment will by no means generate or suppress spins.

As a classical spin model, Schulman[10] stipulated his spinning object as "mechanically indistinguishable from" a spherical top (for which no spatial extension is implied) of $SU(2)$ symmetry. The propagator for such an object covers all the states for both integral and half-integral spins. Indeed, *the $SU(2)$ structure allows the spin quantum number to have both integral and half-integral values irrespective of whether the top has a magnetic moment or not.*

Accepting the $SU(2)$ structure as the key, one might argue that there are no logical grounds for the assertion that symmetry of the freely spinning classical object is $SU(2)$ in lieu of $SO(3)$, but that the presence of the magnetic interaction would induce the $SU(2)$ behavior. However, we can easily rebut this. The transformation (19) can be achieved by a group element of $SO(3)$ as well. Switching on the magnetic interaction amounts to rotating the system about the axis along the direction of the external field. Notice that the state functions for the magnetic top may also be viewed as those of the free top *adiabatically* rotated about the axis of the applied magnetic field,

$$\tilde{\Psi}_{Jmn}(\varphi, \psi, \theta; t) = \sqrt{2J+1}\, e^{im\varphi} e^{in\psi}\, d^J_{mn}(\theta) \tag{39}$$

with the energy spectrum of a spherical top, $E_J = J(J+1)\,\hbar^2/(2I)$ shifted only by a constant $Ig^2B^2/2$. Thus, a single-valued state function (once the quantum number m is fixed to be an integer) in the rotating frame will remain a single-valued function in the absence of the magnetic field. Even classically, the external magnetic field may be seen as a fictitious phenomenon resulting from a rotation of a coordinate frame.[21] There is no mechanism that makes the magnetic top more qualified than a free top in endowing itself with $SU(2)$ symmetry.

In conclusion, we wish to emphasize that the magnetic top has $SU(2)$ symmetry as the free top does and that it is, as a classical model of spin, equivalent to the free top, as discussed by Schulman, at least from the path integral aspect.

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